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Anyons emerging from fermions with conventional two-body interactions

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Abstract

Emergent anyons are the key elements of the topological quantum computation and topological quantum memory. We study a two-component fermion model with conventional two-body interaction in a fine-tuned external field and show that several subsets in the low-lying excitations obey the same fusion rules as those of the toric code model. Those string-like (or domain wall) excitations whose energy congregates in a small spatial region (a wall) may be thought of as quasiparticles which, in a given subset, obey mutual semionic statistics . We show how to peel off one of such subsets from other degenerate subsets and manipulate anyons in cold dipolar Fermi atoms or cold dipolar fermionic heteronuclear molecules in optical lattices by means of the established techniques.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Anyons are long wistful objects in two-dimensional condensed matter systems [1, 2]. An interest to explore anyons is currently renewed because of the potential application of anyons in topological quantum computation and topological quantum memory [3, 4]. The research mainly focuses on two topics: non-Abelian fractional quantum Hall states [5–7] and particular lattice spin models, e.g. the Kitaev toric code model [3], Levin-Wen model [8] and Kitaev honeycomb-lattice spin model [9].

Experimentally, exciting, manipulating and detecting Abelian anyons have been suggested or tried for the toric code model [10, 11] and for the insulating phase of the Kitaev honeycomb-lattice model [12]. Although these non-trivial attempts are interesting, reliable evidence for the existence of anyons is still lacking since neither the toric code model nor the honeycomb-lattice

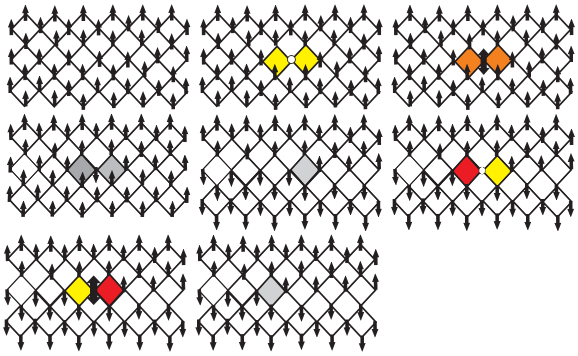

Figure 1. The ground state (taking $|G_\uparrow\rangle$ as an example) and the low-lying excitations in a set of decoupled Ising spin chains which form a square lattice. From left to right and up to down, they are $|G_\uparrow\rangle$, $|\mathcal{H}_{i_a}\rangle$, $|\mathcal{D}_{i_a}\rangle$, $|\mathcal{F}_{i_a}\rangle$, $|\mathcal{W}_P\rangle$, $|\mathcal{W}^h_{P,P'}\rangle$, $|\mathcal{W}^d_{P,P'}\rangle$ and $|\mathcal{W}_{P'}\rangle$. Up and down arrows label the fermion with spin-up and spin-down. The empty circle is unoccupied site and up–down arrow is double occupied. The white plaquette P'' has $(G^s_{P''}, G^{\bar{s}}_{P''}) = (1, 1)$, the yellow has $(-1, 1)$, the red has $(1, -1)$ and the grey has $(-1, -1)$.

spin model is easy to be realized owing to the unconventional interactions between constitution particles, e.g. cold atoms [13]. Very recently, the existence of non-Abelian anyons has been evidenced in the fractional quantum Hall system with the filling factor $\nu = 5/2$ [14].

A widely interesting question is as follows: Do we have a simple system with conventional two-body interactions where some kinds of elementary excitations are anyons? One example is a p-wave paired fermionic model in a square lattice, which is induced from the ground state of the Kitaev honeycomb model and then the non-trivial statistics of some types of elementary excitations are expected [15, 16]. Here we study a two-component fermion system which is a set of decoupled Ising chains[4]. These chains form a two-dimensional square lattice with each chain along the horizontal diagonal direction (see figure 1). The fermions within a chain interact with on-site repulsive and nearest neighbor attractive potentials. Nussinov and Ortiz [18] have found that the decoupled Ising chains are of the same spectrum as the toric code model and Liven–Wen model. However, the ground state in general is not degenerate and the topological order is trivial in this system.

In general, a string-like excitation (see figure 1) with a length L is an excitation with a higher energy of order const $\cdot L$. When the in-chain coupling constants are specially chosen with respect to the external field so that const $= 0$, this string-like excitation becomes a low-lying one and turns out to obey anyonic statistics. We will see that the low-lying excitations of the model in such a special choice of the parameters are classified by several kinds of closed subsets. One kind of them is local, including a single hole, a double occupant and a spin-flip. Another two kinds consist of a local fermion and two string-like excitations. The fusion rules and exchange phase factor in the latter two are exactly the same as those of the excitations in the toric code model. We note that although these string-like excitations are non-local in space, their excitation energy is still limited in one or two sites. In this sense, we can think of these string-like excitations as quasiparticles. The statistics of the string-like excitations themselves are bosonic while being mutual semionic.

We find that the ground state in this model is stable for a dipole–dipole long-range interaction if inter-chain couplings are negligible. Thus, this system can be realized in

[4] This conventional interacting fermion model can be mapped into a simple spin model in a honeycomb lattice. See [17].

cold dipolar Fermi atoms, e.g. rare-earth atoms of Ytterbium [19], or the cold fermionic heteronuclear molecules like ^{40}K^{87}Rb [20], in an optical lattice. With respect to the ground state degeneracy, any subset of the excitations is accompanied by many energetically degenerate subsets. We discuss how to peel off a subset from other degenerate subsets and manipulate anyons by means of the established cold atom and molecular techniques.

2. A two-component fermion model in a square lattice

We consider a simple Hamiltonian for two-component fermions in a square lattice (figure 1):

$$H = -\sum_{\langle ij\rangle_{hd},s} J_s(2n_{s,i}-1)(2n_{s,j}-1) + U\sum_i(2n_{\uparrow,i}-1)(2n_{\downarrow,i}-1) + V\sum_i(2n_{\uparrow,i}-1), \tag{1}$$

where $n_{s,i} = c^\dagger_{s,i}c_{s,i}$ with $c_{s,i}$ being annihilation operators of (pseudo)spin-s fermions. The symbol $\langle ij\rangle_{hd}$ means that the sum is over nearest neighbors along the horizontal diagonals of squares. This is a set of decoupled Ising chains along the horizontal diagonals of squares. We restrict to the nearest neighbor attractive interaction while on-site is repulsive, i.e. $J_s > 0$ and $U > 0$. When $V \neq 0$, the ground state is ferromagnetic and is not degenerate, i.e. all spins are either up for $V < 0$ or down for $V > 0$. The low-lying excitations are trivial. Fermionic excitations are a hole and a double occupant while the bosonic excitation is spin-flip. Therefore, for general parameters, there is no non-trivial topological order in this system.

A topologically non-trivial excitation is a string-like excitation. When $V \neq 0$, the string-like excitation with the length L (see figure 1) is high energy which is of the order of $|V|L$. However, if we tune the external potential V to vanish, the ground state becomes highly degenerate and non-trivial string-like low-lying excitations emerge. In the following, we focus on $V = 0$. In the case when the ground states of this Hamiltonian are 2^n-fold degenerate, i.e. every individual chain is ferromagnetic, for a set of spins $\{s_1,\ldots,s_a,\ldots,s_n\}$,

$$|G_{\{s\}}\rangle = \prod_{a=1}^{n}|G_{s_a}\rangle = \prod_{a,i_a} c^\dagger_{s_a,i_a}|0\rangle, \tag{2}$$

where n is the number of chains and i_a is the site index in the ath chain. We first restrict our study to the open boundary condition. The low-lying excitations above a given ground state $|G_{\{s\}}\rangle$, for a given site i_a, read (see figure 1 for a given ground state $|G_\uparrow\rangle$ which will be defined later)

$$\begin{aligned}
\mathcal{H}_{i_a}|G_{\{s\}}\rangle &= \left(c^\dagger_{s_a,i_a} + c_{s_a,i_a}\right)|G_{\{s\}}\rangle = c_{s_a,i_a}|G_{\{s\}}\rangle,\\
\mathcal{D}_{i_a}|G_{\{s\}}\rangle &= \left(c^\dagger_{\bar{s}_a,i_a} + c_{\bar{s}_a,i_a}\right)|G_{\{s\}}\rangle = c^\dagger_{\bar{s}_a,i_a}|G_{\{s\}}\rangle,\\
\mathcal{F}_{i_a}|G_{\{s\}}\rangle &= \mathrm{i}\mathcal{H}_{i_a}\mathcal{D}_{i_a}|G_{\{s\}}\rangle = \mathrm{i}c_{s_a,i_a}c^\dagger_{\bar{s}_a,i_a}|G_{\{s\}}\rangle,\\
\mathcal{W}_P|G_{\{s\}}\rangle &= \prod_{i'_b\leqslant i_a}\mathcal{F}_{i'_b}|G_{\{s\}}\rangle = \prod_{i'_b\leqslant i_a}\mathrm{i}c_{s_b,i'_b}c^\dagger_{\bar{s}_b,i'_b}|G_{\{s\}}\rangle,\\
\mathcal{W}^h_{P,P'}|G_{\{s\}}\rangle &= \prod_{i'_b<i_a}\mathcal{F}_{i'_b}\mathcal{H}_{i_a}|G_{\{s\}}\rangle = \prod_{i'_b<i_a}\mathrm{i}c_{s_b,i'_b}c^\dagger_{\bar{s}_b,i'_b}c_{s_a,i_a}|G_{\{s\}}\rangle,\\
\mathcal{W}^d_{P,P'}|G_{\{s\}}\rangle &= \prod_{i'_b<i_a}\mathcal{F}_{i'_b}\mathcal{D}_{i_a}|G_{\{s\}}\rangle = \prod_{i'_b<i_a}\mathrm{i}c_{s_b,i'_b}c^\dagger_{\bar{s}_b,i'_b}c^\dagger_{\bar{s}_a,i_a}|G_{\{s\}}\rangle,
\end{aligned} \tag{3}$$

where P and P' denote two plaquettes on the right and left of i_a, respectively. $\bar{s} =\downarrow$ ($\uparrow$) if $s =\uparrow$ ($\downarrow$). The order of sites is defined by $i'_b < i_a$ if i'_b is on the left of i_a for $b = a$ or i'_b

Table 1. Fusion rules of excitations.

	$\mathcal{H}_{i_a}$	$\mathcal{D}_{i_a}$	$\mathcal{F}_{i_a}$	$\mathcal{W}_P$	$\mathcal{W}^h_{P,P'}$	$\mathcal{W}^d_{P,P'}$	$\mathcal{W}_{P'}$
$\mathcal{H}_{i_a}$	I	$\mathcal{F}_{i_a}$	$\mathcal{D}_{i_a}$	$\mathcal{W}^d_{P,P'}$	$\mathcal{W}_{P'}$	$\mathcal{W}_P$	$\mathcal{W}^h_{P,P'}$
$\mathcal{D}_{i_a}$	$\mathcal{F}_{i_a}$	I	$\mathcal{H}_{i_a}$	$\mathcal{W}^h_{P,P'}$	$\mathcal{W}_P$	$\mathcal{W}_{P'}$	$\mathcal{W}^d_{P,P'}$
$\mathcal{F}_{i_a}$	$\mathcal{D}_{i_a}$	$\mathcal{H}_{i_a}$	I	$\mathcal{W}_{P'}$	$\mathcal{W}^d_{P,P'}$	$\mathcal{W}^h_{P,P'}$	$\mathcal{W}_P$
$\mathcal{W}_P$	$\mathcal{W}^d_{P,P'}$	$\mathcal{W}^h_{P,P'}$	$\mathcal{W}_{P'}$	I	$\mathcal{D}_{i_a}$	$\mathcal{H}_{i_a}$	$\mathcal{F}_{i_a}$
$\mathcal{W}^h_{P,P'}$	$\mathcal{W}_{P'}$	$\mathcal{W}_P$	$\mathcal{W}^d_{P,P'}$	$\mathcal{D}_{i_a}$	I	$\mathcal{F}_{i_a}$	$\mathcal{H}_{i_a}$
$\mathcal{W}^d_{P,P'}$	$\mathcal{W}_P$	$\mathcal{W}_{P'}$	$\mathcal{W}^h_{P,P'}$	$\mathcal{H}_{i_a}$	$\mathcal{F}_{i_a}$	I	$\mathcal{D}_{i_a}$
$\mathcal{W}_{P'}$	$\mathcal{W}^h_{P,P'}$	$\mathcal{W}^d_{P,P'}$	$\mathcal{W}_P$	$\mathcal{F}_{i_a}$	$\mathcal{H}_{i_a}$	$\mathcal{D}_{i_a}$	I

is in a chain lower than the chain with i_a. $\mathcal{H}, \mathcal{D}, \mathcal{F}$ create a hole, a double occupant and a spin-flip. $\mathcal{W}$, $\mathcal{W}^d$ and $\mathcal{W}^h$ create a half-infinite string of spin-flips, a spin-flip string with a double occupant and a spin-flip string with a hole, respectively, since the spins of fermions at sites $i'_b < i_a$ are flipped from their ground state configuration while those at $j_c > i_a$ keep in their ground state configuration. The excitation energies of these local and string-like excitations in turn are $4J_{s_a} + 2U$, $4J_{\bar{s}_a} + 2U$, $4J_\uparrow + 4J_\downarrow$, $2J_\uparrow + 2J_\downarrow$, $2J_\uparrow + 2J_\downarrow + 2U$ and $2J_\uparrow + 2J_\downarrow + 2U$, respectively. The finite energies of the string-like excitations are located at a one or two sites means that they are deconfined and can be thought as quasiparticles. These excitations are highly degenerate due to the degeneracy of the ground states. For example, if $\{s_1, \ldots, s_a \ldots, s_n\} \to \{s_1, \ldots, \bar{s}_a \ldots, s_n\}$, $\mathcal{H} \leftrightarrow \mathcal{D}$ and $(\leqslant, <) \to (\geqslant, >)$ in (3). One can also flip spin in other chains to get a new degenerate excitation.

3. Fusion rules

Note that $\mathcal{O}^2 = I$ (the identity operator) for $\mathcal{O} = \mathcal{H}, \mathcal{D}, \mathcal{F}, \mathcal{W}, \mathcal{W}^d, \mathcal{W}^h$. The fusion rules of these excitations are given in table 1. The fusion rules of the closed subset $\{I, \mathcal{D}_{i_a}, \mathcal{W}_P, \mathcal{W}^h_{P,P'}\}$ $(\text{or}\{I, \mathcal{H}_{i_a}, \mathcal{W}_P, \mathcal{W}^d_{P,P'}\})$ are exactly the same as the fusion rules in the Kitaev toric code model if we identify $\mathcal{D}_{i_a}, \mathcal{W}_P$ and $\mathcal{W}^h_{P,P'}$ as ψ, e and m, respectively [3, 9]. The subset $\{I, \mathcal{F}_{i_a}, \mathcal{W}^h_{P,P'}, \mathcal{W}^d_{P,P'}\}$ has similar fusion rules, but $\mathcal{F}$ is bosonic. $\{I, \mathcal{H}_{i_a}, \mathcal{D}_{i_a}, \mathcal{F}_{i_a}\}$ is also a closed subset and has similar fusion rules but with two fermions $(\mathcal{H}, \mathcal{D})$ and one boson $(\mathcal{F})$.

4. Discrete gauge symmetry

The conserved quantities are simply $Q_{s,i} = 2n_{s,i} - 1$ (in fact, they are $n_{s,i}$), which have eigenvalues ± 1 at each site. This generates a $Z_2 \times Z_2$ gauge symmetry. For a plaquette P, one can label the plaquette by $A^{s_a}_P = (2n_{i_a,s_a} - 1)(2n_{j_a,s_a} - 1)$ for a pair of nearest neighbors (i_a, j_a) in the ath chain, which also have eigenvalues ± 1. Obviously, the ground state is of all $A^s_P = 1$. The excitations are also of all $A^s_{P''} = 1$ except for the right plaquette P and left plaquette P' of i_a where $(A^{s_a}_P, A^{\bar{s}_a}_P, A^{s_a}_{P'}, A^{\bar{s}_a}_{P'}) = (-1, 1, -1, 1)$ for $\mathcal{H}_{i_a}$, $(1, -1, 1, -1)$ for $\mathcal{D}_{i_a}$, $(-1, -1, -1, -1)$ for $\mathcal{F}_{i_a}$, $(1, 1, -1, -1)$ for $\mathcal{W}_P$, $(\pm 1, \mp 1, \mp 1, \pm 1)$ for $\mathcal{W}^h_{P,P'}$ and $(\mp 1, \pm 1, \pm 1, \mp 1)$ for $\mathcal{W}^d_{P,P'}$. To distinguish the latter two, one uses $(Q_{s_a,i_a}, Q_{\bar{s}_a,i_a}) = (-1, -1)$ for $\mathcal{W}^h_{P,P'}$ and $(1, 1)$ for $\mathcal{W}^d_{P,P'}$. In this sense, these string-like excitations are also called $Z_2 \times Z_2$ vortices.

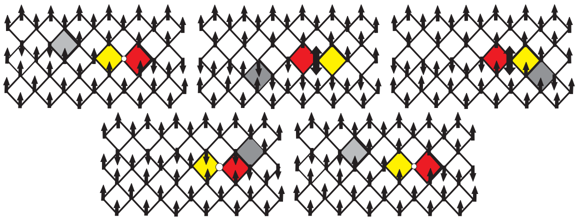

Figure 2. An example that two mutual anyons circle one another. Most of motions of the energy centers of a quasiparticle may be realized by local operations. From the left to right, the upper line to lower, a $\mathcal{W}$ circles around a $\mathcal{W}^d$. However, when one quasiparticle moves across the chain that another quasiparticle lives on, one has to flip many spins as shown in this figure.

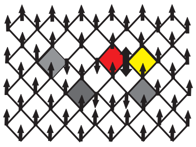

Figure 3. Two pairs of domain walls in a torus by identifying left and right boundaries and the upmost chain and the lowest chain.

5. Periodic boundary conditions and global degeneracy of the ground state

There are two global conserved quantities $A_\uparrow = \prod_{i\in C_a}(2n_{i_a,\uparrow} - 1)(2n_{j_a,\uparrow} - 1)$ and $A_\downarrow = \prod_{i\in C_a}(2n_{i_a,\downarrow} - 1)(2n_{j_a,\downarrow} - 1)$, where C_a denotes a chain. If we put this model to a torus and consider a periodic boundary condition, we have two constraints $A_{\uparrow,\downarrow} = \prod_{i\in C}(2n_{i_a,-1})^2 = 1$ because of $n_{N_a,s} = n_{1_a,s}$ and $(2n_{i,s} - 1)^2 = 1$ where N_a is the number of the sites of C_a. According to Kitaev [3], we can count the global degeneracy of the ground states, i.e. $2^{n-(n-2)} = 4$. Therefore, the topological degeneracy of the present model is the same as that in the toric code model. This is consistent with a straightforward observation for two-decoupled Ising rings [18].

A string-like excitation may be thought of as a spin flipping domain wall, a topological defect (see figure 1, latter four). The open boundary condition allows an odd number of domain walls in a given chain while the periodic boundary condition restricts the wall number to even in a given chain. In the open boundary condition, two walls as ends of a string, e.g. $\cdots\uparrow\uparrow\circ\downarrow\downarrow\cdots$ and $\cdots\downarrow\downarrow\uparrow\uparrow\cdots$, may locate at different chains so that one can circle around another as shown in figure 2. For the periodic boundary condition, the domain walls appear pairwise, e.g. as shown in figure 3. Such a pair of walls cannot exchange. However, two walls in different pairs can exchange.

6. Statistics in a given subset

We now study the statistics of the string-like excitations within a subset. The local excitations $\mathcal{H}, \mathcal{D}$ are the fermionic while $\mathcal{F}$ is bosonic. For a string-like excitation, the site of its domain wall, i_a, may be used to label its 'position', e.g. $\mathcal{W}_P = \mathcal{W}_{i_a}$, etc, because their energies are stored near these 'positions'. $\mathcal{W}$ is bosonic because it is a string of $\mathcal{F}$. $\mathcal{W}^{d,h}$ themselves are bosonic, e.g.

$$\mathcal{W}_1^h\mathcal{W}_2^h = (\mathcal{H}_1)(\mathcal{H}_1\mathcal{D}_1\mathcal{H}_2) = (\mathcal{H}_1\mathcal{D}_1\mathcal{H}_2)(\mathcal{H}_1) = \mathcal{W}_2^h\mathcal{W}_1^h.$$

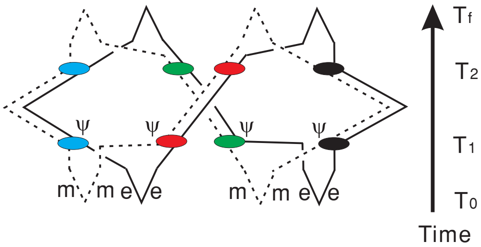

Figure 4. Two pairs of m (dashed lines) and two pairs of e (solid lines) created from the ground state at T_0; they fuse into four fermions ψ (colored ellipses) at T_1; then they split back to four pairs at T_2 and annihilate to the ground state at T_f. The green and red fermions exchange as the time evolution, which leads to the wave function, differs by a minus sign from that of no exchange. The arrow indicates the time direction.

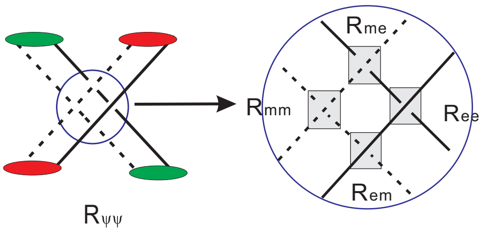

Figure 5. The fermion exchange $R_{\psi\psi}$ divided into exchanges between the same type of string-like excitations, R_{ee} and R_{mm}, and double exchange e and m, $R_{me}R_{em}$, which is equal to moving e (or m) around m (or e).

Since $\mathcal{W}$ and $\mathcal{W}^{d,h}$ are distinguishable, the exchange between them does not make sense. However, because $\mathcal{W}\mathcal{W}^{h,d}$ fuse into a fermion while themselves being bosons, when $\mathcal{W}$ circles around $\mathcal{W}^{h,d}$ or vice versa, a minus sign is acquired. In general, this fact can be proved by applying the consistent conditions, i.e. the pentagon and hexagon equations [21]. For the present case, one can take Kitaev's graphical proof [9]. For simplicity, denote $\{I, \mathcal{H}, \mathcal{W}, \mathcal{W}^d\}$ (or$\{I, \mathcal{D}, \mathcal{W}, \mathcal{W}^h\}$) $= \{I, \psi, e, m\}$. Since $e^2 = m^2 = I$, we can create two pairs of e and two pairs of m from a given ground state at an initial time T_0 (see figure 4). These excitations move along the paths as shown in figure 4. At certain time T_1, they fuse into four fermions ψ. As time flies, while blue and black fermions stay alone, green and red fermions exchange their positions. Finally, at T_2, the fermions split into m and e pairs which, at the end (T_f), fuse back into the ground state. As two fermions exchange, this process contributes a minus sign $R_{\psi\psi} = -1$ to the ground state compared to a process without fermion exchange. Now, examine this process in string-like excitation exchanges. As shown in figure 5, this fermion exchange corresponds to four exchanges: R_{em}, R_{ee}, R_{mm} and R_{me}. That is,

$$R_{me}R_{ee}R_{mm}R_{em} = R_{me}R_{em} = R_{\psi\psi} = -1, \tag{4}$$

since $R_{ee} = R_{mm} = 1$ as e and m themselves are bosonic. The minus sign when e and m doubly exchange, or equivalently, e encircles m, $R_{me}R_{em} = -1$, proves that the mutual statistics between e and m is semionic. Note that for the subset $\{I, \mathcal{F}, \mathcal{W}^d, \mathcal{W}^h\}$, since $\mathcal{F} = \mathcal{W}^d\mathcal{W}^h$ is bosonic, $R_{\mathcal{W}^d\mathcal{W}^h}R_{\mathcal{W}^h\mathcal{W}^d} (=R_{\mathcal{F}\mathcal{F}} = 1)$ is trivial.

Using the operator formalism, the exchange process described by figure 5 for four quasiparticles at $i_a < j_b < k_c < l_d$ is, for example, as follows:

$$\begin{aligned}W_{i_a}^d W_{j_b} W_{k_c}^d W_{l_d} &= W_{i_a-1}\mathcal{D}_{i_a} W_{j_b} W_{k_c-1}\mathcal{D}_{k_c} W_{l_d}\\ &= \left(\prod_{i<i_a}\mathcal{F}_i\right)\mathcal{D}_{i_a}\left(\prod_{j\leqslant j_b}\mathcal{F}_j\right)\left(\prod_{k<k_c}\mathcal{F}_k\right)\mathcal{D}_{k_c}\left(\prod_{l\leqslant l_d}\mathcal{F}_l\right)\\ &= -\left(\prod_{k<k_c}\mathcal{F}_k\right)\mathcal{D}_{k_c}\left(\prod_{l\leqslant l_d}\mathcal{F}_l\right)\left(\prod_{i<i_a}\mathcal{F}_i\right)\mathcal{D}_{i_a}\left(\prod_{j\leqslant j_b}\mathcal{F}_j\right)\\ &= -W_{k_c}^d W_{l_d} W_{i_a}^d W_{j_b},\end{aligned} \tag{5}$$

since $\{\mathcal{D}_i,\mathcal{D}_j\} = 2\delta_{ij}$, $[\mathcal{F}_i,\mathcal{F}_j] = 2\delta_{ij}$, $\{\mathcal{F}_i,\mathcal{D}_i\} = 0$, and $[\mathcal{F}_i,\mathcal{D}_j]_{i\neq j} = 0$. On the other hand, figure 5 corresponds to a series of exchanges R_{em}, $R_{ee}R_{mm}$, R_{me} which, in the operator formalism, are given by $W_{i_a}^d W_{j_b} W_{k_c}^d W_{l_d} \to W_{i_a}^d W_{k_c}^d W_{j_b} W_{l_d} \to W_{k_c}^d W_{i_a}^d W_{l_d} W_{j_b} \to W_{k_c}^d W_{l_d} W_{i_a}^d W_{j_b}$. Since the second right arrow does not make a sign change, the sign exchange appearing in equation (5) comes from $R_{me}R_{em}$ as expected.

7. Peeling off a subset from degeneracy

So far, we only say that string-like excitations obey mutual semionic statistics but not call them anyons or semions. The reason for this is there are many degenerate string-like states which are not in the same subset. For example, flipping any chain's spin for $\mathcal{W}_{\mathcal{P}}$ results in a degenerate excitation with $\mathcal{W}_{\mathcal{P}}$ but already out of the subset of $\mathcal{W}_{\mathcal{P}}$. In this sense, these excitations cannot be identified as individual quasiparticles.

To peel off a designed subset, we need to set barriers between individual degenerate ground states without changing their energies. To control the electron spin of each individual chain is not easy. However, it becomes possible in a cold atom (molecule) system because the 'spin' we are studying actually labels the different hyperfine states of atoms in the cold atom context . Once an atom is in a given hyperfine state, local fluctuations from the environment cannot switch it to others. Therefore, we can peel off a given string-like excitation from others by preparing the ground state. For example, we can apply a magnetic field so that only the atoms with a given hyperfine state are loaded into the lattice and then turned off the magnetic field after the system is stable at the ground state. A global ferromagnetic ground state $|G_\uparrow\rangle = |G_{\{\uparrow,\uparrow,\ldots,\uparrow\}}\rangle$ is prepared. All excitations in (3) can then be prepared by creating, annihilating the fermions or changing fermions from $\uparrow$ to $\downarrow$ hyperfine states by means of the recently developed stimulated Raman spectroscopy or photoemission spectroscopy technique[22]. Here, annihilating and creating a fermion does not mean removing fermions from or reloading them into lattice sites. They can be turned into other hyperfine states which are almost not coupled to the 'spin' $\uparrow$ and $\downarrow$ hyperfine states or reverse. The string-like excitations may be controllably prepared. We may merely prepare excitations in a given subset, and they are barricaded from their degenerate states. The string-like excitations obeying mutual simonic statistics in this subset are now identified as *mutual semions*.

Recently, a realization of this model in superconducting circuits has been proposed [23]. It is also a possible way to peel off a semionic subset.

8. Cold fermions with dipole–dipole interaction

For cold fermions in an optical lattice, the off-site interaction between the cold atoms can be induced by their diploe–dipole interaction. Recently, the degenerate Fermi gases of rare-earth atoms of Ytterbium (Yb) have been obtained [19]. They are a possible candidate to be a practical system for our model because the fermionic isotopes ^{171}Yb and ^{173}Yb are stable in nature and their metastable state ^{3}P$_2$ has a large magnetic dipole moment $3\mu_B$. Deeply bound cold fermionic heteronuclear molecules have a much larger dipole moment, e.g. the electric dipole moment of ^{40}K^{87}Rb in its absolute bound ground state is 0.3 ea_B [20]. Load the fermions in an optical lattice and polarize all dipoles along the horizontal diagonal of squares by using an external field. The interaction potential is $V_d(\mathbf{r}, \mathbf{r}') = d^2 \frac{1-3\cos^2\Theta}{R^3}$ where Θ is the angle between $R = \mathbf{r} - \mathbf{r}'$ and d (the dipole moment of an atom). The interactions along the diagonal become attractive. The repulsive interaction is restricted in the region with $\Theta > \Theta_c$ for $\cos^2\Theta_c = 1/3$. The interacting Hamiltonian can be written as

$$V = -\sum_{\langle ij\rangle_{hd},s} |V_{ij,s}| n_{s,i} n_{s,j} - \sum_{\langle\langle ij\rangle\rangle_{hd},s} |V_{ij}| n_{s,i} n_{s,j} - \sum_{ij,0<\Theta<\Theta_c,s} |V_{ij}| n_{s,i} n_{s,j} + \sum_{ij,\Theta>\Theta_c,s} V_{ij} n_{s,i} n_{s,j}, \tag{6}$$

where $\langle\langle ij\rangle\rangle_{hd}$ denotes the sum along the horizontal diagonal other than the nearest neighbors. It is very easy to stabilize the ground state because one may increase the distance between the horizontal chains or adjust the optical lattice potentials so that the inter-chain couplings become weak.

Strictly speaking, the anyons emerging from these dipolar particle systems are logarithmic confinement in thermodynamic limit, i.e. $E_{\text{pair}} - E_g \propto \ln L$ as $L \to \infty$ for L the distance between a pair of the string-like excitations. This weak divergence in a practical optical lattice may be abided, e.g. if the short-range model we proposed has $E_{\text{pair}} - E_g \sim 1$, this logarithmic excitation energy is 4.5 for $L = 50$. Even for $L = 1000$, this energy only increases about 15 times.

9. Conclusion

In conclusion, we have proved that it is possible to find non-trivial mutual anyonic statistics in a fermionic system with conventional two-body interaction under a fine-tuned external field. We showed that the domain wall or string-like quasiparticles are deconfined in these fine-tuned parameters and obey the same fusion as those in the toric code model. By means of a graphitic method, we repeated Kitaev's proof for the mutual semionic statistics of those quasiparticles but our proof is more understandable. We also verified this mutual semionic statistics in an operator formalism. How to peel up a subset of semionic quasiparticles from many degenerate states and how to manipulate them were discussed. We also pointed out that these anyons can be realized in cold fermions with dipole–dipole interaction.

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